MATH1010 I/J University Mathematics, 2015-16

§ 1 Preliminaries

1.1 Notations

Set : collection of objects (elements)

$\subseteq$: subset

$\in$: belongs to

Example 1.1.1

$S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T, or $S \subseteq T$.

That means every element in S is also an element in T.

Notations often used in this course :

$\mathbb{N}$: set of all natural numbers

$\mathbb{Q}$: set of all rational numbers

$\mathbb{R}$: set of all real numbers

ϕ : empty set, i.e. $\phi = \{ \}$ Nothing

$[a, b]$: set of all real numbers x such that $a \leq x \leq b$

(a, b) : set of all real numbers x such that $a < x < b$

$[a, \infty)$: set of all real numbers x such that $a \leq x$

Example 1.1.2

Set of all positive even integers

$= \{2, 4, 6, \ldots\}$

$= \{2m : m \in \mathbb{N}\}$

i.e. this set consists of elements of the form $2m$ such that $m \in \mathbb{N}$.

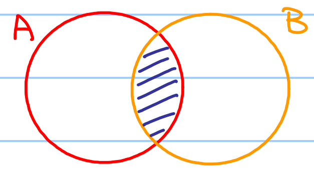
Exercise 1.1.1

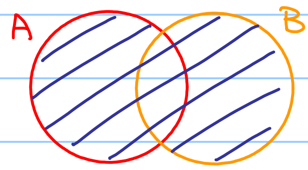
Set of all positive odd integers = ? (How to describe?)

Answer : $\{2m-1 : m \in \mathbb{N}\}$

Set Operations :

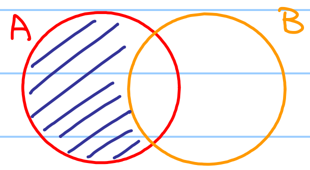
Let A, B be two sets.

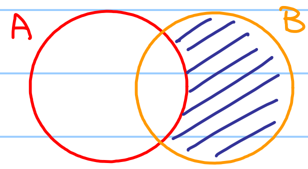




Intersection : $A \cap B$

Union : $A \cup B$





Relative complement of B in A : $A \setminus B$

Relative complement of A in B : $B \setminus A$

Example 1.1.3

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3\}$

- $A \cap B = \{2\}$ $\quad A \cap C = \phi$
- $A \cup B = \{1,2,3\}$
- $A \setminus B = \{1\}$ $\quad B \setminus A = \{3\}$

Example 1.1.4

$\mathbb{R} \setminus \{2\}$: set of all real numbers except 2

(Caution : We cannot write $\mathbb{R} \setminus 2$!)

Example 1.1.5

Solve $x^2 > 1$.

$\therefore \quad x > 1$ or $x < -1$

OR : $x \in (-\infty, -1) \cup (1, \infty)$

OR : $x \in \mathbb{R} \setminus [-1, 1]$

$\forall$: for all

$\exists$: there exists (at least one)

$\exists!$: there exists unique

$\Rightarrow$: implies

$\Leftrightarrow$: if and only if (equivalent to)

s.t. : such that

Example 1.1.6

$\forall\, y \in (0,\infty),\ \exists\, x \in \mathbb{R}$ s.t. $x^2 = y$.

$\downarrow$ translate

For all positive real number y, there exists (at least one) real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ or $-\sqrt{y}$)

$\forall\, y \in (0,\infty),\ \exists!\, x \in (0,\infty)$ s.t. $x^2 = y$.

$\downarrow$ translate

For all positive real number y, there exists unique positive real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ only)

Example 1.1.7

If $x > 0$, $y = \sqrt{x} \Rightarrow y^2 = x$

but $y^2 = x \not\Rightarrow y = \sqrt{x}$ (Why?)

Example 1.1.8

In a ΔABC,

$\angle ABC = 90^\circ \Rightarrow AB^2 + BC^2 = AC^2$ (Pyth. thm.)

$AB^2 + BC^2 = AC^2 \Rightarrow \angle ABC = 90^\circ$ (Converse of Pyth. thm.)

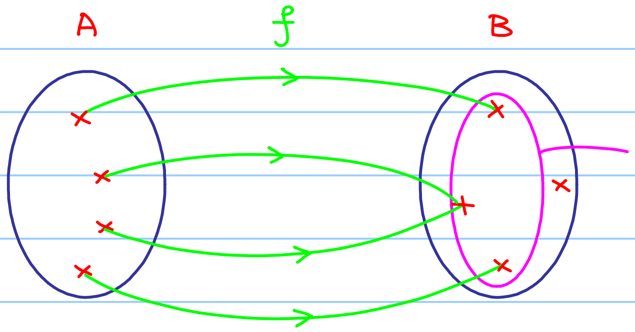
If both statements are true, we say

$\angle ABC = 90^\circ$ if and only if $AB^2 + BC^2 = AC^2$

and denote it by $\angle ABC = 90^\circ \Leftrightarrow AB^2 + BC^2 = AC^2$

1.2 Functions

Function : A function is a rule that assigns to each element in a set A exactly one element in a set B.



set A : domain (input)

set B : range (output)

image(f) $\subseteq$ B : image of f

image(f) $= f(A) := \{ f(x) \in B : x \in A \}$

defined by

A function f from A to B is denoted by $f : A \to B$

Example 1.2.1

If 1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ image(f) $= [0, \infty)$

2) $f : [-1, 2) \to \mathbb{R}$ defined by $f(x) = x^2$ image(f) $= [0, 4)$

Example 1.2.2

If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4$

$f(-3) = (-3)^2 + 4 = 13$

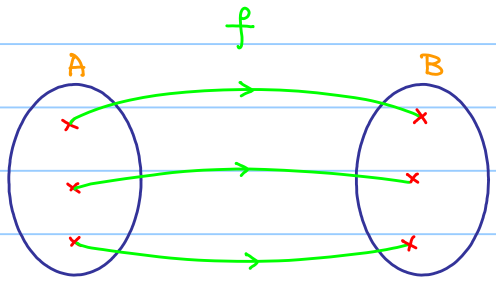
input   output

OR write : $y = x^2 + 4$

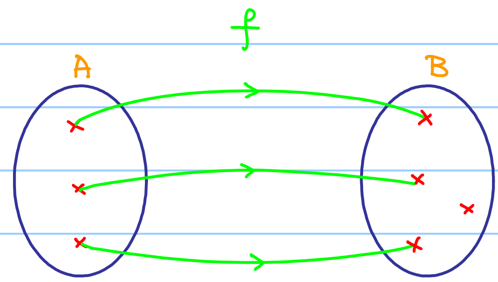
dependent variable   independent variable

Injective and Surjective Functions :

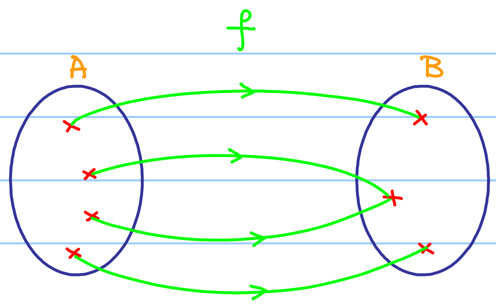
Intuitive idea :

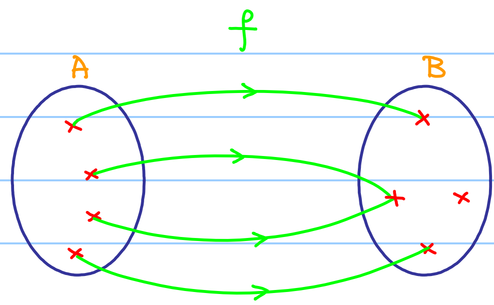


injective + surjective



injective but NOT surjective





surjective but NOT injective

injective : every $y \in \text{image}(f)$ comes from exactly one $x \in A$

surjective : every $y \in B$ comes from one $x \in A$

Definition 1.2.1

Let $f : A \to B$ be a function.

1) f is said to be an injective function if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

(Explanation : Once the output are the same, the inputs must be the same !)

2) f is said to be an surjective function if

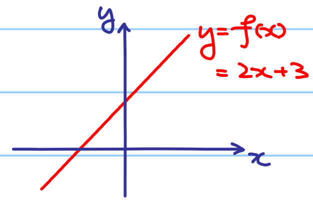
$$\forall y \in B, \exists x \in A \text{ s.t. } f(x) = y \qquad (f(A) = B)$$

If f is both injective and surjective, then it is said to be bijective.

Example 1.2.3

Show $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=2x+3$ is a bijective function.

1) Injective:



$$f(x_1)=f(x_2)$$

$$\Rightarrow \quad 2x_1+3 = 2x_2+3$$

$$\Rightarrow \quad x_1 = x_2$$

$\therefore f$ is injective.

2) Surjective:

Let $y\in\mathbb{R}$,

take $x=\frac{y-3}{2}\in\mathbb{R}$

then $f(x)=f\left(\frac{y-3}{2}\right)$

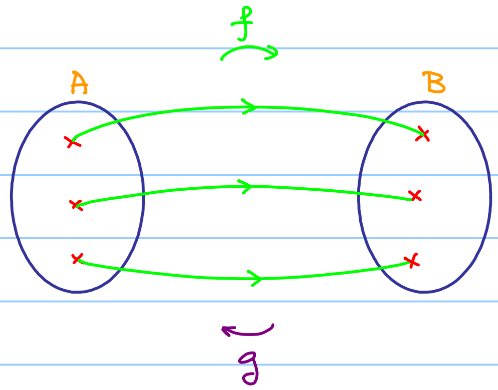
$=2\left(\frac{y-3}{2}\right)+3$

$=y$

$\therefore f$ is surjective.

Inverse of a Function

Intuitive idea:



Definition 1.2.2

Let $f:A\to B$ be a function. If $g:B\to A$ is a function such that

1) $g(f(x))=x \quad \forall x\in A$

2) $f(g(y))=y \quad \forall y\in B$

Then g is said to be an inverse of f.

Fact: 1) Once an inverse of f exists, it is unique, we denote it by f^{-1}.

2) f has an inverse $\Leftrightarrow$ f is bijective.

Example 1.2.4

		injective	surjective
$f: \mathbb{R} \to \mathbb{R}$	defined by $f(x) = \sin x$	×	×
$f: \mathbb{R} \to [-1,1]$	defined by $f(x) = \sin x$	×	✓
$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$	defined by $f(x) = \sin x$	✓	✓

$\therefore$ We can define arcsin function !

$\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

We write $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$, then

$\sin^{-1}(\sin x) = x \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(\sin^{-1} y) = y \quad \forall y \in [-1,1]$

1.3 Sequences of Real Numbers

Example 1.3.1

Let $a_1 = 2$, $a_2 = \pi$, $a_3 = \sqrt{3}$, $\cdots$

OR write as $\{2, \pi, \sqrt{3}, \cdots\}$ (No pattern)

Example 1.3.2

Sequences having patterns.

Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $\cdots$ in general, $a_n = 2^{n-1}$

Let $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $\cdots$ in general, $a_n = \frac{1}{n}$

Let $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $\cdots$ in general, $a_n = (-1)^n$

Example 1.3.3

Recursive sequence.

Let $\{a_n\}$ be a sequence of real numbers defined by $a_1 = 1$ and $a_{n+1} = a_n^2 + 2$ for $n \geq 1$.

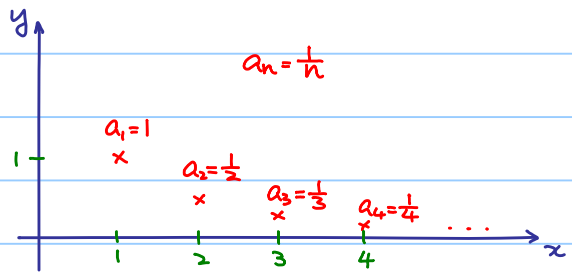
Then $\{a_n\} = \{1, 3, 11, 123, \cdots\}$.

Remark / Definition 1.3.1

A sequence of real numbers $\{a_n\}$ can be regarded as a function $f: \mathbb{N} \to \mathbb{R}$

and $a_n = f(n)$ (i.e. given $n \in \mathbb{N}$, return the n-th entry of the sequence)

A sequence can be understood by the following diagram.



Any observation?

When n is getting larger and larger, a_n is getting closer and closer to 0.